The Complexity of the Evolution of Graph Labelings

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Introduction

A graph is here assumed simple and undirected, $G = (V(G), E(G))$, where

$$E(G) \subseteq \binom{V(G)}{2},$$

the set of all 2-sets of $V(G)$.

Let

- $|V(G)| = n$ is the order of G, and
- $|E(G)| = m$ is the size of G.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the natural numbers.

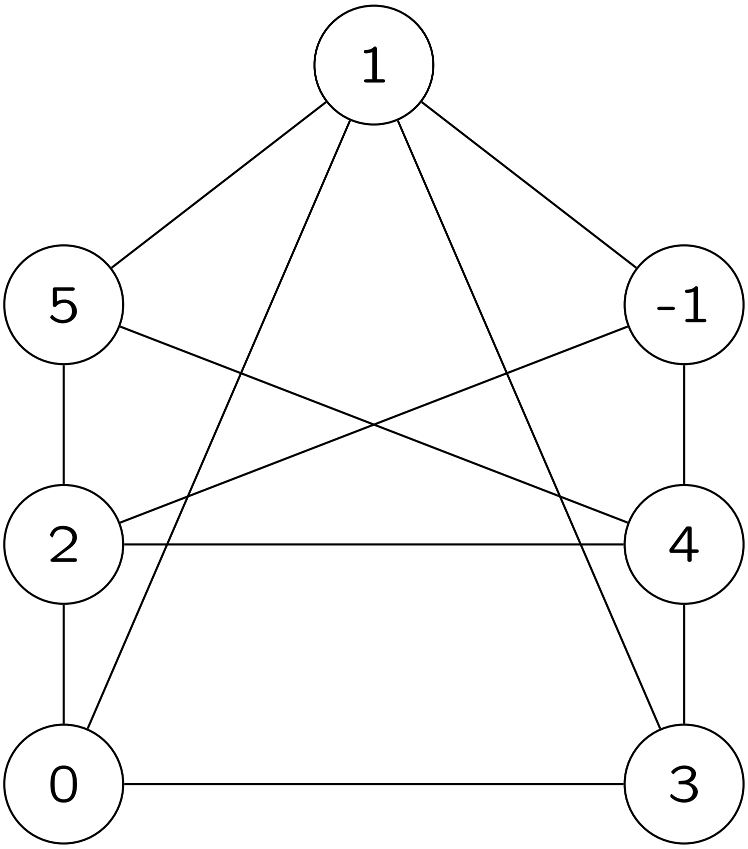
Let $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ be the integers.

A graph labeling is a function

$$f : V(G) \to \mathbb{Z},$$

assigning integer labels to the vertices (or edges, or both!) subject to certain conditions.

We often restrict the labels to be natural numbers in a fixed range.


1
5
-1
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3

Classic Examples of Graph Labelings

(A) Graceful Labelings

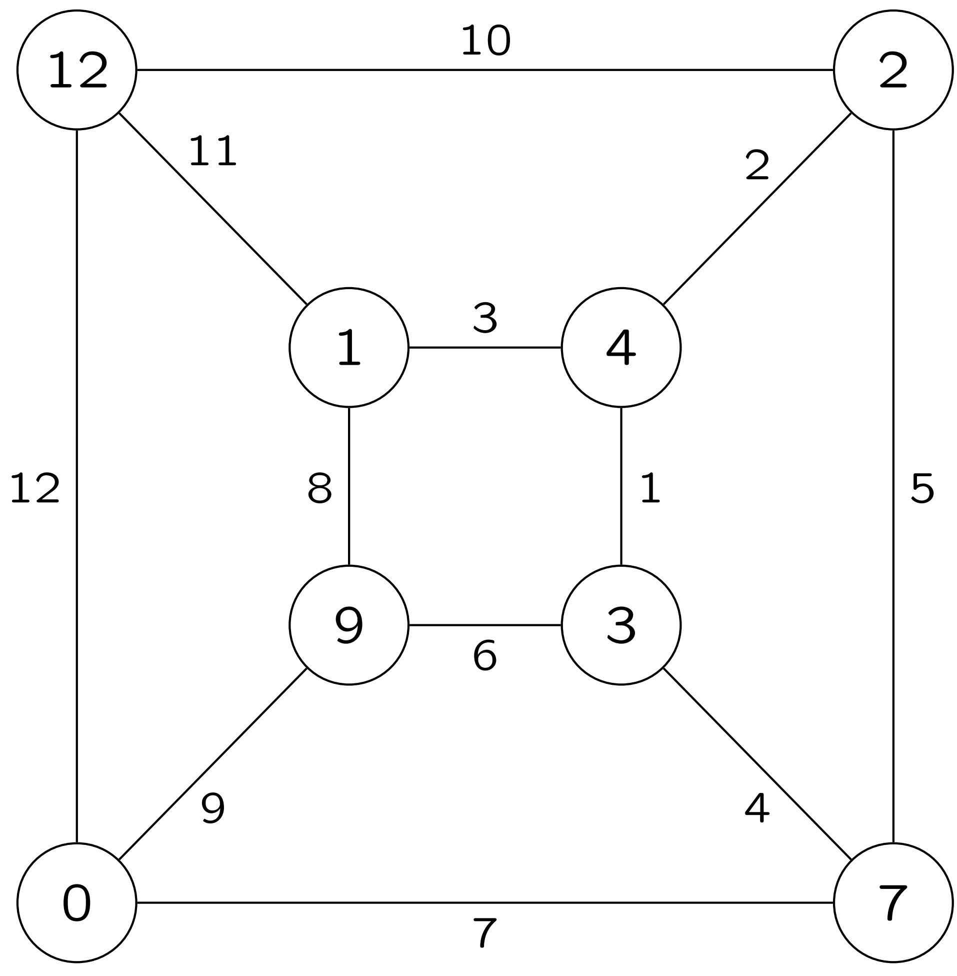
A graph labeling $f : V(G) \to \{0, 1, \ldots, m\}$ is graceful if f is injective and induces an injection

$$E(G) \ni \{u, v\} \mapsto |f(u) - f(v)| \in \mathbb{N}.$$

Example: All paths and rectangular grid graphs are graceful.

Conjecture
Every tree has a graceful labeling.

This conjecture is called the Graceful Labeling Conjecture and attributed to Ringel, Kotzig, & Rosa (1967), and also Von Koch as Von Koch's Conjecture.


12
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11
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7

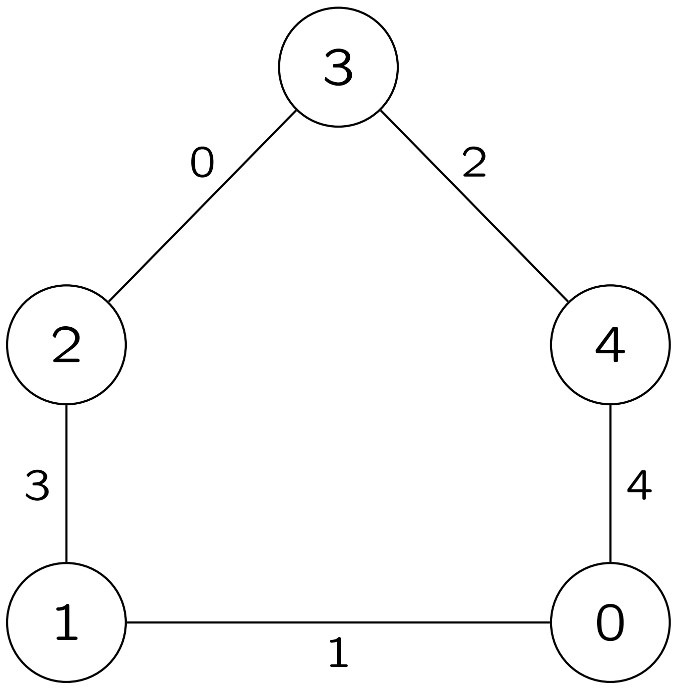
(B) Harmonious Labelings

A graph labeling $f : V(G) \to \mathbb{Z}_m$ is harmonious if f is injective and induces an injection

$$E(G) \ni \{u, v\} \mapsto f(u) + f(v) \in \mathbb{Z}_m.$$

Example: A tree has no harmonious labeling. (Clear by PHP, since $m = n - 1$.)

Example: The cycle C_n on n vertices and edges has a harmonious labeling if and only if n is odd.


3
0
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4
1
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1

(C) Magic Labelings

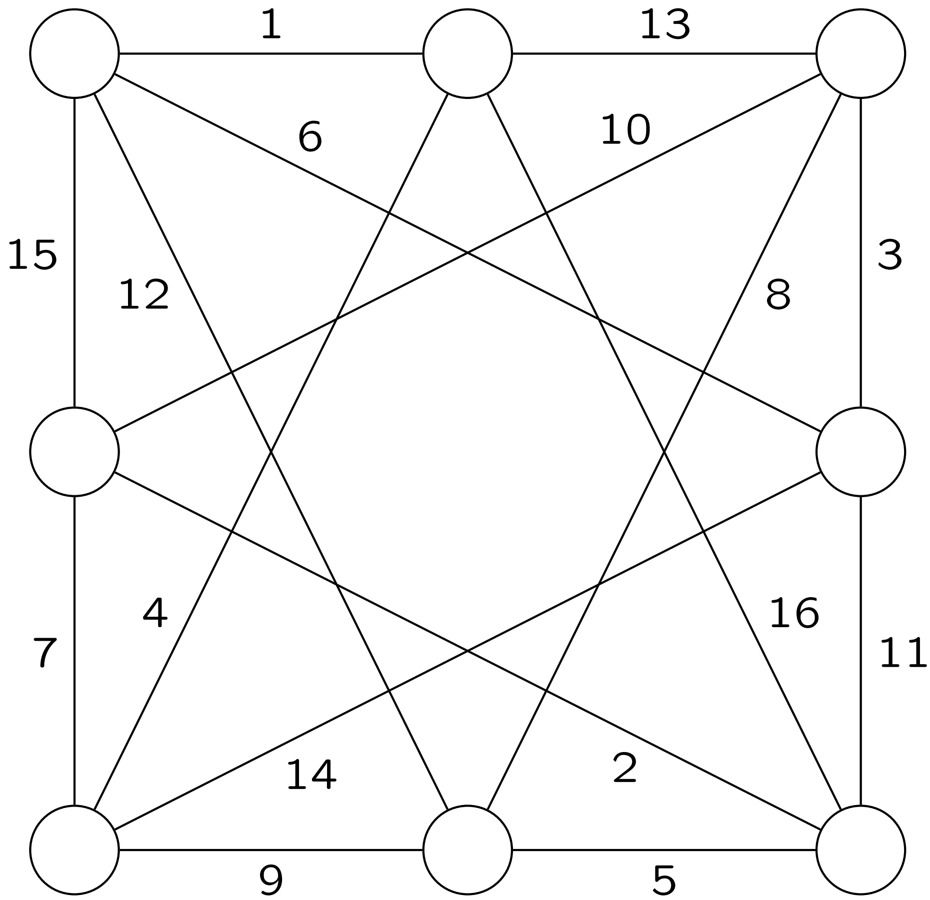
(i) A graph labeling $f : E(G) \to \mathbb{Z}$ such that for any two vertices $u, v \in V(G)$, we have

$$\sum_{e \ni u} f(e) = \sum_{e \ni v} f(e)$$

is called semi-magic.

(ii) A semi-magic labeling f with $f : E(G) \to \mathbb{N}$ an injection is a magic labeling.

The magic labelings are motivated by and generalize the magic squares in number theory.


1
13
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15
12
3
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16
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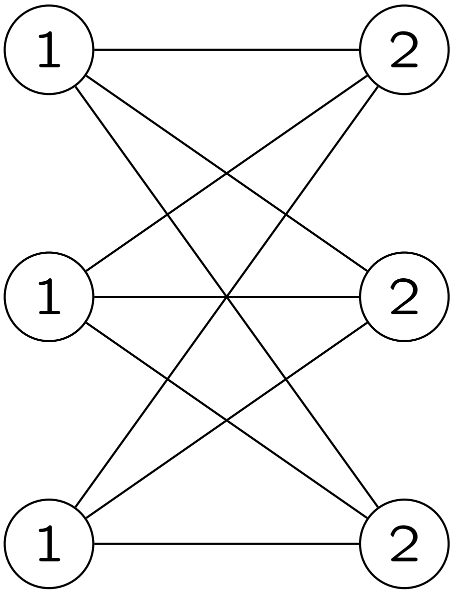
14
2
9
5

(D) Graph Colorings

A graph labeling $f : V(G) \to \mathbb{N}$ such that

$$\{u, v\} \in E(G) \Rightarrow f(u) \neq f(v)$$

is the usual vertex coloring of a graph G.


1
2
1
2
1
2

Questions about Labelings

Typical questions regarding any labeling are:

- ▷ For $G \in \mathcal{G}$ (a given class of graphs) does G have a XXX labeling?
- ▷ For $G \in \mathcal{G}$ and $k \in \mathbb{N}$ does G have a XXX labeling $f : V(G) \to \{1, \ldots, k\}$?
- ▷ What is the smallest such k? And so on.

For a good survey of various graph labelings, applications, open problems, and so on, see

A Dynamic Survey of Graph Labeling,

by Joseph Gallian (revised 2009).

Transformation of Labelings

Related questions involve a given graph G, two labelings L and L', and a collection of specific moves $\mathcal{M}$.

- ▷ Can L be transformed to L' by using moves from $\mathcal{M}$?

- ▷ For a given fixed graph G, what is the least $t = t(G) \in \mathbb{N}$ such that any L can be transformed to any L' using at most t moves from $\mathcal{M}$?

- ▷ For $G \in \mathcal{G}(n)$ (a given class of graphs on n vertices), what is the least $t = t(n) \in \mathbb{N}$ such that any L can be transformed to any L' using at most t moves from $\mathcal{M}$?

Example:

Pancake Flipping Problem

Given a labeling $L : V(P_n) \to \{1, \ldots, n\}$ (that is, a permutation in S_n), and using moves/flips given by

$$\begin{aligned}\mathcal{F}_{pc} = \{&(\ell_1, \ldots, \ell_n) \\ &\mapsto (\ell_1, \ldots, \ell_i, \ell_n, \ell_{n-1}, \ldots, \ell_{i+1})\},\end{aligned}$$

(resembling the flip of a sub-stack of pancakes) can we always obtain the natural right-to-left labeling (resembling the largest pancake at the bottom, next largest pancake on top of that, and so on) using flips from $\mathcal{F}_{pc}$? How many times do we need to flip for a given labeling L?

Example:

Permutations

Given two labelings $L, L' : V(P_n) \to \{1, \dots, n\}$ and the set of flips

$$\mathcal{F} = \{(\dots, \ell_i, \ell_{i+1}, \dots) \mapsto (\dots, \ell_{i+1}, \ell_i, \dots)\}.$$

- ▷ Can we always transform L to L' using flips from $\mathcal{F}$? (Yes, bubble sort!)
- ▷ How many flips from $\mathcal{F}$ are needed?

This is a well-known problem with a nice solution:

- ▷ May assume L' is the labeling $L'(i) = i$.
- ▷ Let $p(L)$ be the number of inversion pairs. That is, (i, j) with $i < j$ and $L(i) > L(j)$.

Observation [Thomas Muir (1882)] *Each consecutive flip from $\mathcal{F}$ reduces or increases $p(L)$ by exactly one.*

Corollary *We need at least $p(L)$ flips from $\mathcal{F}$ to obtain L' from L.*

It is easy to see that we can obtain L' from L by $p(L)$ flips from $\mathcal{F}$.

Proposition *For the path P_n the labeling L can be transformed to L' in t flips from $\mathcal{F}$ if and only if $t \geq p(L, L')$.*

General Graph Relabeling

Now consider a general graph G.

Let $\mathcal{F}_{\mathcal{G}}$ be the set of all label-flips between two adjacent vertices, that is,

$$(\ldots, L(u), L(v), \ldots) \mapsto (\ldots, L(v), L(u), \ldots)$$
$$\Leftrightarrow \{u, v\} \in E(G).$$

1. Let T be a spanning tree of G.

2. Let $u_1, \ldots, u_n$ be the vertex numbers (not labels!) of the Prüfer code order (when leaves of T are deleted during the process of constructing the Prüfer code).

3. The Prüfer code provides a leaf elimination order of T.

Theorem *Let G be a simple connected graph on n vertices, and L and L' given vertex labelings. Then L can be transformed to L' in $t = \frac{n(n-1)}{2}$ flips from $\mathcal{F}_G$.*

Proof:

- ▷ To move $L(u_{i_1}) = L'(u_1)$ from u_{i_1} to u_1 requires at most $n-1$ flips from $\mathcal{F}_G$.

- ▷ To move $L(u_{i_2}) = L'(u_2)$ from u_{i_2} to u_2 requires at most $n-2$ flips from $\mathcal{F}_G$, and so on.

- ▷ We always move along the shortest path in T between u_{i_ℓ} to u_ℓ whose length is $\leq n-\ell$.

- ▷ We use at most $(n-1)+(n-2)+\cdots+1 = \frac{n(n-1)}{2}$ flips. □

NB! $t = \frac{n(n-1)}{2}$ is the best/smallest such number possible for a general graph G on n vertices, since

$$p(n, n-1, \ldots, 1) = \binom{n}{2} = \frac{n(n-1)}{2},$$

so there is a graph (namely, P_n) and labelings L and L' (namely, $L : (n, n-1, \ldots, 1)$ and $L' : (1, 2, \ldots, n)$) that requires $\frac{n(n-1)}{2}$ flips from $\mathcal{F}_{P_n}$.

For most classes $\mathcal{G}(n)$ of graphs on n vertices, the value of $t = t(n)$ is not known.

In particular,

Open Question *What is the smallest $t = t(n)$ such that for any two labelings L and L' of the grid graph, L can be transformed into L' using at most t flips?*

Priviledged Labels

One can also consider privileged labels, where one is only allowed restricted flips, where one of the labels must be privileged.

Example:

The 15-Puzzle [Noyes Chapman (1880's)]

4×4 board, one privileged square, others labeled $1, 2, \ldots, 15$. The goal is to obtain

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & * \end{bmatrix}$$

Observation *The Vertex Labeling Problem with Priviledged Labels is NP-complete.*

Some general questions:

Given a graph G on n vertices, labelings L and L' as injections to $\{1, \ldots, n\}$, $P \subseteq \{1, \ldots, n\}$ a set of privileged labels, and $t \in \mathbb{N}$.

- ▷ Can L be transformed into L' in t or fewer restricted flips?

- ▷ Can L be transformed into L' in t or fewer restricted flips in polynomial time?

- ▷ What are the bounds of $t = t(G; P)$?

By considering the left/right order of the non-privileged labels on an n-path P_n, we have

Observation *Among all connected labeled graphs on n vertices, with $k \in \{0, 1, \ldots, n-2\}$ privileged labels the Vertex Relabeling With Privileged Labels Problem is unsolvable.*

Similarly, by considering the clockwise/counter-clockwise orientation of the non-privileged labels on the n-cycle C_n, we have

Observation *Among all 2-connected labeled graphs on n vertices, with $k \in \{0, 1, \ldots, n-3\}$ privileged labels the Vertex Relabeling With Privileged Labels Problem is unsolvable.*

The case where all but exactly two labels are privileged has a nice solution!

A key lemma:

Lemma *Among vertex labeled trees, which are not paths, with exactly two non-privileged labels, any two labels can be swapped using restricted flips.*

Using the key lemma plus other observations, we can provide a complete description to obtain the following:

Theorem *Among all connected labeled graphs G on $n \geq 4$ vertices, with exactly $n-2$ privileged labels the Vertex Relabeling With Privileged Labels Problem is solvable (in polynomial time!) if and only if G is not a path.*

Open Problems

- ▷ Study other types of mutation functions where, for example, labels along an entire path are mutated, or where labels can be reused.

- ▷ In the parallel setting, compute the sequence of mutations required for the transformation of one labeling into another. The parallel time for computing the sequence could be much smaller than the sequential time to execute the mutation sequence.

  One result of interest in this direction is the problem of given a labeled graph, a *prescribed flipping sequence*, and two designated labels l_1 and l_2 are l_1 and l_2 flipped? A prescribed flipping sequence is an ordering of edges in which each succeeding edge's labels may be flipped if and only if neither of its labels has already been flipped. This problem is *NC*-equivalent to the Lexicographically First Maximal Matching Problem, and so *CC*-complete.

▷ For various classes of graphs determine the probability of one labelings evolving naturally into another. Such an evolution of a labeling could be used to model mutation periods.

▷ Study the properties of the graphs of all labelings. In this graph all labelings of a given graph are vertices and two vertices are connected if they are one mutation apart. Other conditions for edge placement may also be worthwhile to examine.

- Determine if there is a version of the Edge Relabeling with Privileged Labels Problem that is *NP*-complete.

- Define the *cost of a mutation sequence* to be the sum of the weights on all edges that are mutated. Determine mutation sequences that minimize the cost of evolving one labeling into another. Explore other cost functions.

Thanks for coming ⌣̈